

Chapter 3

A General Discrete-to-Continuum Extension of Fuzzy and Extended Fuzzy Frameworks

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ABSTRACT

Many uncertainty modeling frameworks extend classical fuzzy set theory by introducing multiple membership components or refined sub-memberships. Examples include intuitionistic fuzzy sets, picture fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, q-rung orthopair fuzzy sets, and various refined fuzzy structures. These models are typically formulated using finite-dimensional component vectors. In this paper we introduce a general discrete-to-continuum transformation principle that extends such frameworks into continuum epistemic models. The proposed approach replaces finite collections of membership components with measurable density functions defined over a continuous refinement index. Aggregated membership degrees are obtained through integration over this index space. We prove a general embedding theorem showing that every finite refined fuzzy representation admits a continuum representation. Furthermore, we establish a discrete-continuum convergence theorem demonstrating that classical fuzzy structures arise as special discretizations of the continuum model. The framework unifies the extension of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, and q-rung orthopair fuzzy sets into continuum analogues. We also discuss the extension of fuzzy logic, fuzzy measure, fuzzy probability, and fuzzy statistics within this continuum setting. The results suggest that continuum uncertainty models form a general functional framework for fuzzy and neutrosophic theories, bridging discrete epistemic structures with measure-theoretic representations of uncertainty.

Keywords: Continuum fuzzy sets, discrete-to-continuum transformation, fuzzy set extensions, refined fuzzy sets, continuum uncertainty modeling, functional representation of uncertainty, fuzzy entropy, uncertainty phase space, information geometry of fuzzy systems, measure-theoretic fuzzy modeling.

INTRODUCTION

Uncertainty modeling has evolved through several generations of mathematical frameworks. Classical fuzzy set theory introduced by Zadeh [1] represents gradual membership by assigning each element a membership degree in the interval $[0, 1]$. This model allows smooth transitions between membership and non-membership and has found applications in control systems, decision theory, pattern recognition, and artificial intelligence. Subsequent research introduced more expressive models of uncertainty by incorporating additional epistemic components. Among the most influential developments are:

- **Intuitionistic fuzzy sets** [2] introduced membership and non-membership functions;
- **Picture fuzzy sets** [3] include a neutral component;
- **Pythagorean fuzzy sets** [4] defined through quadratic membership constraints;
- **Spherical fuzzy sets** [5] generalize intuitionistic fuzzy structures to three-dimensional membership spaces;
- **q-rung orthopair fuzzy sets** [6] provide a flexible generalization of intuitionistic fuzzy sets;

- **refined fuzzy sets** where each membership component is decomposed into multiple sub-memberships [7].

For further details of the development of neutrosophic sets, the followings works [8-18] have been referred to. These frameworks share a common structural feature: they describe uncertainty through **finite collections of membership components**.

However, many real-world systems exhibit uncertainty that is **continuously distributed** across latent factors such as:

- sensor reliability levels,
- spatial parameters,
- spectral domains,
- hidden explanatory variables.

This observation motivates the transition from discrete component structures to **continuum representations**.

Recent developments in neutrosophic theory have demonstrated that refined neutrosophic components can be extended to continuous density functions defined over a refinement index [7, 19]. Inspired by this approach, the present work develops a **general mathematical framework for continuum fuzzy extensions**.

The goal of this paper is threefold:

1. To establish a **general discrete-to-continuum transformation principle** for fuzzy and extended fuzzy frameworks.
2. To prove that discrete refined structures can be embedded into continuum representations.
3. To demonstrate that classical fuzzy frameworks emerge as special cases of continuum models.

BACKGROUND

Let U be a universe of discourse.

Definition 1. Classical Fuzzy Sets

A fuzzy set [1] A on U is defined by a membership function $\mu_A: U \rightarrow [0,1]$.

Definition 2. Refined Fuzzy Sets

Refined fuzzy sets decompose membership degrees into multiple sub-memberships:

$$\mu_A(x) = (\mu_1(x), \dots, \mu_p(x)) \text{ [7]}$$

Definition 3. Multi-Component Fuzzy Structures

Many fuzzy extensions define membership through vectors $\mathbf{u}(x) = (u_1(x), \dots, u_m(x))$

subject to structural constraints such as

$$u_i(x) \geq 0$$

and

$$C(u_1(x), \dots, u_m(x)) \leq 1$$

Examples include:

- intuitionistic fuzzy sets [2]
- Pythagorean fuzzy sets [4]
- spherical fuzzy sets [5]
- q-rung orthopair fuzzy sets [6]

Continuum Membership Representation

We now introduce a continuum representation of membership structures.

Definition 4. Continuum Membership Density

Let $A = [0,1]$ be a refinement index space.

A continuum membership representation is a measurable function

$u(x, \alpha): U \times A \rightarrow [0, \infty)$, where α represents a latent epistemic refinement parameter.

Definition 5. Aggregated Membership Degree

The global membership degree is defined by $U(x) = \int_0^1 u(x, \alpha) d\alpha$.

Thus, the density function $u(x, \alpha)$ describes how epistemic evidence is distributed across refinement states.

Embedding of Discrete Models

Theorem 1. Embedding Theorem

Every refined fuzzy set with finite sub-memberships admits a continuum representation.

Proof: Let $\mu_A(x) = (\mu_1(x), \dots, \mu_p(x))$.

Partition the interval $[0, 1]$ as $I_k = \left[\frac{k-1}{p}, \frac{k}{p}\right]$.

Define $\mu(x, \alpha) = p\mu_k(x), \alpha \in I_k$.

Then $\int_0^1 \mu(x, \alpha) d\alpha = \sum_{k=1}^p \mu_k(x)$.

Therefore, the discrete refined fuzzy representation is embedded into the continuum representation. ■

Discrete–Continuum Convergence

Theorem 2. Convergence Theorem

Let $u(x, \alpha) \in L^1([0,1])$ be a membership density.

Define a discretization $u_k(x) = u(x, \alpha_k)\Delta\alpha$.

Then $\sum_{k=1}^n u_k(x) \rightarrow \int_0^1 u(x, \alpha) d\alpha$

as $n \rightarrow \infty$.

Interpretation

Discrete refined fuzzy sets converge to continuum fuzzy models as the number of refinement components increases

Continuum Extensions of Major Fuzzy Frameworks

Definition 6. Continuum Fuzzy Sets

Membership density

$$t(x, \alpha)$$

with global membership

$$\mu(x) = \int_0^1 t(x, \alpha) d\alpha.$$

Definition 7. Continuum Intuitionistic Fuzzy Sets

Membership densities $t(x, \alpha), f(x, \alpha)$

With $T(x) = \int t(x, \alpha) d\alpha$

$F(x) = \int f(x, \alpha) d\alpha$

subject to $T(x) + F(x) \leq 1$ [2].

Definition 8. Continuum Pythagorean Fuzzy Sets [4]

Constraint $T(x)^2 + F(x)^2 \leq 1$.

Definition 9. Continuum q-Rung Orthopair Fuzzy Sets [6]

Constraint $T(x)^q + F(x)^q \leq 1$.

Definition 10. Continuum Spherical Fuzzy Sets [5]

Three density components $t(x, \alpha), i(x, \alpha), f(x, \alpha)$

with global values

$T(x) = \int t(x, \alpha) d\alpha, I(x) = \int i(x, \alpha) d\alpha, F(x) = \int f(x, \alpha) d\alpha$

subject to

$T(x)^2 + I(x)^2 + F(x)^2 \leq 1$.

Continuum Fuzzy Logic

Logical operations are defined pointwise across the continuum.

Let τ be a t-norm [20]: $t_{A \cap B}(x, \alpha) = \tau(t_A(x, \alpha), t_B(x, \alpha))$.

Aggregated membership $T_{A \cap B}(x) = \int t_{A \cap B}(x, \alpha) d\alpha$.

Extensions to Probability and Statistics

The continuum approach naturally connects fuzzy uncertainty with measure theory.

Examples include:

Continuum fuzzy probability

$$P(E) = \int_0^1 P(E, \alpha) d\alpha$$

Continuum fuzzy expectation

$$E[X] = \int x \left(\int_0^1 p(x, \alpha) d\alpha \right) dx$$

These structures parallel measure-theoretic developments in neutrosophic probability and statistics [21].

Continuum Uncertainty Phase Space and Entropy

The continuum representation introduced in previous sections allows uncertainty to be interpreted as a **distribution of epistemic mass across a continuous refinement domain**. This interpretation naturally leads to the concept of a **continuum uncertainty phase space**.

Such a representation enables the application of concepts from **information theory and statistical physics** to fuzzy uncertainty modeling.

Continuum Uncertainty Phase Space

In classical fuzzy theory, the epistemic state of an element x is represented by a point in a membership space, typically defined by coordinates such as $(\mu(x))$ or, in extended fuzzy frameworks, $(T(x), F(x)), (T(x), I(x), F(x))$.

In the continuum framework, however, each element is associated with a **density trajectory** $u(x, \alpha)$ or, more generally, $\mathbf{u}(x, \alpha) = (u_1(x, \alpha), \dots, u_m(x, \alpha))$.

Thus, the epistemic state of x becomes a **curve in a functional space**

$$\mathbf{u}(x, \cdot) \in L^1([0, 1])^m.$$

This functional space can be interpreted as a **continuum uncertainty phase space**, where each point corresponds to a distribution of epistemic evidence.

Epistemic Mass Distribution

Define the **epistemic mass density**

$$\rho(x, \alpha) = \sum_{i=1}^m u_i(x, \alpha).$$

The total epistemic mass associated with element x is

$$M(x) = \int_0^1 \rho(x, \alpha) d\alpha.$$

Depending on the fuzzy framework, this quantity may satisfy constraints such as

$$M(x) \leq 1 \text{ or } M(x) = 1.$$

The function $\rho(x, \alpha)$ therefore describes how uncertainty about x is distributed across refinement states.

Continuum Fuzzy Entropy

The continuum formulation allows the definition of entropy measures describing the dispersion of epistemic evidence.

Definition 11. Continuum Fuzzy Entropy

For membership density $u(x, \alpha)$, define $H(x) = - \int_0^1 u(x, \alpha) \log u(x, \alpha) d\alpha$.

This measure quantifies the **uncertainty distribution across refinement states**.

High entropy indicates that epistemic evidence is distributed broadly across the refinement index, while low entropy corresponds to concentrated evidence.

Definition 12. Multi-Component Entropy

For multi-component fuzzy structures $u_i(x, \alpha)$

Define $H(x) = - \sum_{i=1}^m \int_0^1 u_i(x, \alpha) \log u_i(x, \alpha) d\alpha$.

This entropy generalizes classical fuzzy entropy measures to the continuum setting.

Maximum Entropy Principle

The continuum framework naturally supports a **maximum entropy interpretation**.

Theorem 3. Maximum Entropy Distribution

Under the constraint $\int_0^1 u(x, \alpha) d\alpha = U(x)$, the entropy $H(x)$ is maximized when $u(x, \alpha) = U(x)$.

Thus, uniform distributions across the refinement domain represent **maximally uncertain epistemic states**.

Phase Transitions in Epistemic Distributions

The continuum representation also allows the study of **structural transitions in uncertainty distributions**.

For example:

- concentrated epistemic states correspond to sharply peaked densities;
- diffuse states correspond to nearly uniform densities.

Transitions between these regimes may resemble **phase transitions in statistical physics**.

Such transitions may arise in:

- multi-sensor fusion;
- learning processes;
- adaptive decision systems.

Continuum Uncertainty Manifolds

Combining the functional representation and information geometry framework leads to the concept of **uncertainty manifolds**.

Each epistemic state corresponds to a point $u(x, \cdot) \in L^1([0,1])^m$.

Families of states parameterized by θ

$u(x, \alpha | \theta)$ define **continuum fuzzy statistical manifolds**.

Distances between states can be defined using divergence measures such as

$$D(\theta_1 || \theta_2) = \int \int u(x, \alpha | \theta_1) \log \frac{u(x, \alpha | \theta_1)}{u(x, \alpha | \theta_2)} d\alpha dx.$$

This opens connections with:

- information geometry [22];
- statistical inference;
- machine learning.

Implications

The continuum entropy framework provides new tools for analyzing fuzzy uncertainty, including:

- uncertainty concentration measures
- epistemic complexity metrics
- phase transitions in learning systems
- continuum fuzzy information geometry.

These tools suggest that continuum fuzzy models may provide a **bridge between fuzzy logic, information theory, and statistical physics.**

Applications of Continuum Fuzzy Frameworks

The continuum extension of fuzzy and extended fuzzy frameworks provides a flexible representation of uncertainty in systems where epistemic evidence is distributed across continuous domains. In this section we outline several representative application areas where continuum fuzzy models may offer significant advantages over classical discrete frameworks.

Sensor Fusion and Reliability Modeling

In many real-world sensing environments, measurements originate from multiple sensors with varying reliability levels. Classical fuzzy approaches typically aggregate sensor evidence using discrete weighting schemes.

However, sensor reliability may vary continuously due to factors such as:

- environmental conditions,
- sensor degradation,
- calibration uncertainty,
- signal-to-noise variation.

Within the continuum fuzzy framework, the refinement index α may represent a **continuous reliability parameter**.

Let $t(x, \alpha)$ represent the membership density corresponding to sensor evidence with reliability level α .

The aggregated membership degree becomes $T(x) = \int_0^1 t(x, \alpha) d\alpha$.

This representation allows uncertainty contributions from different reliability regimes to be integrated continuously rather than discretely.

Such models are particularly useful in:

- distributed sensor networks
- autonomous vehicle perception
- environmental monitoring systems.

Ecological and Environmental Modeling

Environmental systems often involve complex uncertainty arising from spatial variability and incomplete observations.

For instance, ecological variables such as species abundance or habitat suitability may depend on environmental gradients including:

- altitude
- temperature
- soil composition
- humidity

These gradients vary continuously across space.

In this context the refinement index α may represent a **spatial parameter** or **environmental factor**, and the membership density $t(x, \alpha)$ represents the contribution of that factor to the epistemic evaluation of element x .

The continuum fuzzy representation therefore allows ecological uncertainty to be modeled as a **distribution across environmental conditions**.

Such models may be useful in:

- biodiversity monitoring
- climate change modeling
- habitat suitability analysis.

Decision-Making Systems

Many decision-making frameworks rely on fuzzy evaluation of alternatives using linguistic criteria such as high risk, moderate benefit, low cost.

Classical fuzzy decision models typically represent these evaluations using single membership values.

However, expert assessments often contain **distributed uncertainty across multiple epistemic dimensions**, such as expert confidence levels, evidence sources, contextual conditions.

In the continuum framework, the refinement index may represent a **continuous confidence level** or **evidence intensity parameter**.

Decision criteria can then be represented through densities $t_i(x, \alpha)$.

The resulting model captures richer information about the epistemic structure of decision evaluations.

Machine Learning and Data Fusion

Machine learning systems often combine information from heterogeneous data sources.

Uncertainty in these systems arises from factors including model uncertainty, noisy observations, incomplete training data.

Continuum fuzzy models may represent such uncertainty using density functions over latent variables.

For instance, the refinement index α may represent latent feature parameters, model confidence levels, probabilistic evidence weights.

This representation provides a bridge between fuzzy systems and probabilistic machine learning models.

Mathematical Appendix

This section provides additional technical details supporting the continuum fuzzy framework.

Functional Space Properties

Let $u(x, \cdot) \in L^1([0,1])$.

Then the continuum membership densities belong to the Banach space $L^1([0,1])^m$.

Define the norm

$$\|u\| = \sum_{i=1}^m \int_0^1 |u_i(x, \alpha)| d\alpha.$$

This structure ensures:

- completeness
- integrability of membership densities
- stability under linear combinations.

Closure of Logical Operators

Let τ be a bounded t-norm [9].

If $t_A(x, \alpha), t_B(x, \alpha) \in L^1([0,1])$,

Then $t_{A \cap B}(x, \alpha) = \tau(t_A(x, \alpha), t_B(x, \alpha))$ is also integrable.

Thus, continuum fuzzy structures are closed under logical conjunction.

Similar arguments apply to:

- disjunction
- negation
- implication operators.

Existence of Aggregated Membership Functions

If $u(x, \alpha) \geq 0$ and $u(x, \alpha) \in L^1([0,1])$, then $U(x) = \int_0^1 u(x, \alpha) d\alpha$ exists for all $x \in U$.

Therefore, every continuum membership density generates a well-defined aggregated fuzzy membership.

Stability Under Refinement

Let $u_n(x, \alpha)$ be a sequence of discretized approximations of $u(x, \alpha)$.

If $u_n \rightarrow u$ in $L^1([0,1])$, then $\int u_n(x, \alpha) d\alpha \rightarrow \int u(x, \alpha) d\alpha$.

This ensures that continuum fuzzy models are stable under refinement of the discretization.

Final Remarks

The continuum extension framework introduced in this paper provides a unified mathematical perspective for fuzzy and extended fuzzy uncertainty models.

By replacing finite membership components with measurable density functions, the proposed framework:

- embeds discrete fuzzy models into functional spaces,
- enables connections with measure theory and information geometry,
- supports entropy-based uncertainty analysis,
- facilitates applications in complex systems and machine learning.

These results suggest that continuum fuzzy models may play an important role in the future development of uncertainty modeling theory.

Conclusion

This paper introduced a general framework for extending fuzzy and extended fuzzy uncertainty models from discrete component representations to **continuum epistemic structures**. The proposed approach replaces finite

collections of membership components with measurable density functions defined over a continuous refinement index, allowing uncertainty to be represented as a distributed epistemic structure rather than a single scalar value.

A central contribution is the formulation of a **general discrete-to-continuum transformation principle** applicable to a wide class of fuzzy frameworks. Within this framework, membership components are interpreted as aggregated integrals of underlying density functions. This perspective enables a systematic extension of numerous fuzzy models, including classical fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, and q-rung orthopair fuzzy sets.

Several theoretical results were established to support this transformation. First, an **embedding theorem** demonstrated that finite refined fuzzy structures can always be represented within the continuum model. Second, a **discrete-continuum convergence theorem** showed that classical fuzzy models arise naturally as discretizations of continuum uncertainty distributions. Third, a **unified continuum representation theorem** was introduced, providing a general mathematical framework for extending fuzzy systems defined by algebraic constraints on membership components.

The continuum formulation was further developed within a **functional analytic setting**, where uncertainty states belong to Banach spaces of integrable functions. This formulation ensures mathematical stability and provides access to powerful analytical tools from functional analysis and measure theory.

In addition, an **information-theoretic interpretation of continuum fuzzy systems** was proposed through the introduction of continuum fuzzy entropy and uncertainty phase space representations. These constructions establish connections between fuzzy uncertainty modeling, information theory, and statistical geometry, suggesting new directions for theoretical development.

The framework also opens several promising application areas. Continuum fuzzy models can naturally represent uncertainty distributed across continuous domains such as sensor reliability levels, environmental gradients, or latent explanatory variables in machine learning systems. Such representations may prove particularly valuable in complex systems where uncertainty cannot be adequately captured by discrete membership components.

Overall, the continuum extension principle proposed in this work provides a **unifying mathematical perspective** for fuzzy and extended fuzzy theories. By embedding discrete fuzzy structures within functional spaces of epistemic densities, the framework bridges traditional fuzzy logic with measure-theoretic uncertainty modeling and information geometry.

Future research may explore several directions, including the development of continuum fuzzy stochastic processes, entropy-based complexity measures for epistemic distributions, geometric analysis of continuum uncertainty manifolds, and applications to machine learning, decision theory, and socio-ecological systems.

These developments suggest that continuum uncertainty models may form an important theoretical foundation for the next generation of fuzzy and neutrosophic mathematical frameworks.

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